Power variation and variation index

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Content

Abstract

This report lists simple results about the relationship between power variation (including the standard notions of total variation and quadratic variation as special cases) and variation index, which is often used as a measure of volatility. Variation index is defined for a given sequence of partitions.

Variation and variation index

The p-variation of $\omega \in \Omega_1$ (continuous and $\omega(0) = 0$) is defined as

$$var_p(\omega) = \sup_{0=t_0<t_1<\cdots<t_n=1} |\omega(t_i) - \omega(t_{i-1})|^p$$

The variation index is the borderline value

$$vi(\omega) = \inf\{p | var_p(\omega) < \infty\} = \sup\{p | var_p(\omega) = \infty\}.$$

Power variation

A partition of $[0, 1]$ is a finite sequence of numbers
$0 = t_0 < t_1 < \cdots < t_m \leq 1$ ($t_k = \infty$ for $k > m$);
$\pi = (\pi^0, \pi^1, \pi^2, \dots)$ is a nested sequence of partitions.
The nth approximation along π is

$$V_n(t) = A_t^{n,p,\pi} = \sum_{k=1}^{\infty} \left| \omega(\pi_k^n \wedge t) - \omega(\pi_{k-1}^n \wedge t) \right|^p .$$

If $A^{n,p,\pi}(t)$ converges (uniformly?) as $n \to \infty$, then its limit $V(t) = A^{p,\pi}(t)$ is called power p-variation of ω along π.

Relationship set

The relationship set of $\omega \in \Omega_1$ is the set of all $(p_1, p_2) \in [1, \infty) \times [1, \infty)$ for which there exists ω such that:

- ω has nontrivial (non-zero) power p_1-variation along π;
- the variation index of ω is p_2.

Riemann and Lebesgue dyadic partition sequence

Riemann dyadic sequence is $\pi = (\pi^n)$ where $\pi_k^n = k2^{-n}$ $(k = 1, \dots, 2^n)$.

Lebesgue dyadic sequence is defined inductively:

$$\pi_0^n = 0;$$

$$\pi_k^n(\omega) = \inf \left\{ t \in [\pi_{k-1}^n(\omega), 1] | \omega(t) \in \{k2^{-n}\} \setminus \{\omega(\pi_{k-1}^n)\} \right\}$$

Continuous functions

Proposition 1. For continuous functions, the relationship set with respect to Riemann partition π is

$$\{(1,1)\} \cup \{(p_1, p_2) \in (1,\infty) \times (1,\infty] | p_1 \leq p_2\}.$$

Recall that (p_1, p_2) belongs to the relationship set if there is a function f which has power variation p_1 along π, but its variation index of is p_2.

We will use the facts: $p_1 \leq p_2$;
fractional Brownian motion $B_p(t)$ with Hurst index $1/p$ is known to have the property $p_1 = p_2 = p$ a.s.

Example

Define f as the function whose graph connects the points

$$(0,0),(1/2,a_1),(3/4,a_1-a_2),(7/8,a_1-a_2+a_3),\dots$$

by straight lines in this order, where $a_1 > a_2 > a_3 > ... > 0$ ($a_n \to 0$).
What is its power p-variation for $p > 1$?
For the partition $(0,1/2,1)$, $V_1(1) \leq a_1^p + a_2^p = b_1 + a_2^p$;
for the partition $(0,1/4,1/2,3/4,1)$,
$V_2(1) \leq 2^{1-p}a_1^p + a_2^p + a_3^p = b_2 + a_2^p$;
and so on.
In general, $b_n = 2^{1-p}b_{n-1} + a_n^p \to 0$, therefore $V_n(1) \to 0$, which implies $V \to 0$ as V is non-decreasing.

Example for $1 < p_1 \leq p_2 \leq \infty$

Let us combine two functions B_p (Brownian motion of index p) and f (with $a_n = n^{1/p_2}$ for $1 \leq p_2 < \infty$ or $1/log(n+1)$ for $p_2 = \infty$):

$w(t) = B_{p_1}(t)$ for $0 \leq t \leq 1/2$;
$w(t) = w(1/2) + f(2t - 1)$ for $1/2 \leq t \leq 1$.

The first part has variation (p_1, p_1), the second has $(1, p_2)$.
So the whole function has power variation index p_1 along Riemann partition π and the variation index p_2.

Cadlag functions

What is the set of possible (p_1, p_2) is in the case of cadlag (continuous on the right and having limits on the left) functions Ω_2 $(\omega(0) = 0)$ on $[0, 1]$?

We already know that any combination $1 < p_1 \leq p_2$ is possible even for continuous functions.

But for cadlag (unlike continuous) functions the cases $p_1 > p_2$ and $p_1 < 1$ also become non-trivial.
Also, a non-continuous function may have multiple values of p_1 index.

Case $0 < p_2 < p_1$: yes

Recall that p_1 is power p-variation index along Riemann dyadic partition π, p_2 is 'standard' p-variation index.

Consider a piecewise-constant cadlag function with 'gaps' $(-1)^n n^{-1/p_2}$ at points 2^{-n}.

Its standard variation index is p_2 and any number larger than p_2 can be p_1.

Case $p_1 < \min\{1, p_2\}$: no

If this happens then for $p = (p_1 + \min\{p_2, 1\})/2$ power p-variation along π is 0 but the 'standard' p-variation is infinite.

But this is **impossible:** for cadlag functions the standard p-variation for $p < 1$ can not be higher than the power p-variation along π.

The principal reason is: inserting new points into a partition may decrease the total variation for $p > 1$ but not for $p \leq 1$.

Case $p_1 < \min\{1, p_2\}$: no

Indeed, for $p < 1$ variation never decreases after adding new points to the partition. Therefore we can consider two partitions sequences: π and π' where $\pi'_n = \pi_n \cup \rho$ where ρ is a partition along which too high standard p-variation is achieved. Now the question becomes: is it possible that power p-variation along π'_n is higher that along π? Using sequential binary bisection, we can find a specific point $r \in \rho$ around which this effect concentrates. Let $u_n < r < v_n$ be the smallest π_n-interval containing r. If the difference between variations along π' and π really exists then there are r, $\delta > 0$ and n_0 such that $|f(u_n) - f(r)|^p + |f(r) - f(v_n)|^p - |f(u_n) - f(v_n)|^p > \delta$ for any $n > n_0$. But for a cadlag f, $f(v_n)$ tends to $f(r)$ and $f(u_n)$ tends to u, so the whole expression tends to $|u - f(r)|^p - |f(r) - f(r)|^p - |u - f(r)|^p = 0$.

Lebesgue dyadic partition sequence

Recall that Lebesgue dyadic sequence is defined inductively:

$$\pi_0^n = 0;$$

$$\pi_k^n(\omega) = \inf\left\{t \in [\pi_{k-1}^n(\omega), 1] | \omega(t) \in \{k2^{-n}\} \setminus \{\omega(\pi_{k-1}^n)\}\right\}$$

For continuous functions, it is known from Bruneau that $p_1 = p_2$.

For cadlag functions, any $1 \leq p_1 < p_2 \leq \infty$ is possible, this can be derived from Stricker's upper bound for variation.

Lebesgue dyadic partition sequence

The definition is modified for cadlag functions: $\lambda_0^n = D_0^n = 0$;

$$C_k^n(t) = conv\{\omega(\lambda_{k-1}^n), \omega(t)\} \cap \left(\{k2^{-n}\} \setminus D_{k-1}^n(\omega)\right)$$

$$\lambda_k^n(\omega) = \inf\left\{t \in [\lambda_{k-1}^n(\omega), 1] | C_k^n(t) \neq \emptyset\right\}$$

$$D_k^n(\omega) \in \arg \min_{D \in C_k^n(t)} |D - \omega(\lambda_k^n)|$$

Lebesgue dyadic partition sequence

Stricker: for any $q > p$, $\omega \in \Omega$, $\delta > 0$ there exists $c(q, p)$ s.t.

$$var_q(\omega) \leq c \left(\theta^q + \theta^{q-p} \sup_j (L2^{-j})^p M(\omega, L2^{-j}) \right)$$

where $\theta = \sup_{s\in[0,1]} w(s) - \inf_{s\in[0,1]} w(s)$ and $M(\omega, L2^{-j})$ $\left(D(\omega, L2^{-j})\right)$ is the number of upcrossings (downcrossings) of $L2^{-j}$ by ω.

This is true for any $L \geq \theta$, we will use L of the form 2^k.

Lebesgue dyadic partition sequence

Suppose the power p-variation V of $\omega \in \Omega_2$ exists for $p \geq 1$ and $q > p$. By Sticker's result, it is enough to prove that for an integer j the total number of crossings $M(\omega, L2^{-j}) + D(\omega, L2^{-j})$ is bounded above by a linear function (independent of j) of V^j (jth approximation to V) multiplied by 2^{-jp}.

This is achievable by calculating the crossings.

Conclusion

Let p_1 be for power variation index along Riemann dyadic partition sequence π, p_2 for standard variation index.
For continuous functions, either $p_1 = p_2 = 1$ or $1 < p_1 \leq p_2$;
for cadlag functions, any p_1 and p_2 larger than 1 are compatible, but $p_1 < p_2$ is impossible if $p_1 < 1$.

For continuous functions along Lebesgue partition sequence, $1 \leq p_2 \leq p_1$.