April 19, 2019

Q&A for Glenn Shafer

1. **After publishing the book *Probability and Finance: It's Only a Game* in 2001, you and Professor Vovk are now publishing *Game-Theoretic Foundations for Probability and Finance*, which is said to contain fifteen years' worth of further research on this topic. What led to this follow up?**

The game-theoretic picture of probability is a complement, and sometimes an alternative, to the established theory of probability, which has been developed and perfected over centuries. When we published *Probability and Finance*, we knew we could not fully explore this new picture's potential in a single book. So we always expected further editions. We expected that a second edition, involving minor additions and corrections, would come within a few years. As it turned out, further research by ourselves and others quickly led to more substantial developments. We were distracted by other projects and duties, and by the time we could work seriously on a new edition we had learned so much more about the game-theoretic picture that a completely new book was called for.

Probability and Finance introduced the notion of a probability game through concrete examples. The new book adds both a general abstract theory and many more applications. It deals with continuous-time finance in a completely different way, which has proven more productive and is suited to a wider audience.

2. **What were your main objectives during the writing process?**

The book is very ambitious. We want to persuade others who use mathematical probability (mathematicians, computer scientists, theoretical and applied statisticians, engineers, etc.) to use the game-theoretic picture. We also want to draw attention to the light this picture throws on the philosophical meaning of probability and even on its history.

Mathematical probability has a wider range of applications than any other area of applied mathematics. So it's very ambitious to say, "We're going to do this in a different way that will provide insights the usual way doesn't provide." Many people are already familiar with standard ways of understanding mathematical probability, having invested a great deal of time and effort to learning them. These existing approaches now seem intuitive and simple to them. Our goal is to bring out the simplicity and power of the game-theoretic picture so clearly that it will be appreciated both by readers new to mathematical probability and by those expert in the existing theory.

3. **How is game theory related to probability theory and finance theory?**

Game theory has become ubiquitous. It is fundamental to theoretical economics and increasingly important in the other social sciences, in computer science, and even in

pure mathematics. It is a remarkably broad mathematical framework, because games can have any number of players and any number of different rules for how they play, what information they have, and what and how they win. Our book uses only games of a very special kind—*perfect-information* games. In these games, players move in turn and see each others' moves as they play. We show that the notion of probability *emerges from* certain simple perfect-information games.

In a typical game we consider, one player offers bets, another decides how to bet, and a third decides the outcome of the bet. We often call the first player Forecaster, the second Skeptic, and the third Reality. The names express the idea that Skeptic is challenging the validity of Forecaster's offers as a description of the phenomenon represented by Reality. If Skeptic does not succeed in multiplying the money he risks by a large factor, then Forecaster is vindicated. Forecaster's betting offers might be based on a probability distribution for what Reality will do, or they might be less extensive and simpler. We say that an event has small probability (or, sometimes, a small upper probability) if Skeptic has a way of multiplying his money by a lot if it happens.

This is a very new way of connecting probability with game theory. When probability and game theory are mentioned in the same breath, people usually think instead about putting probabilities into games where players do not see others' moves. Some potential readers assumed, from the title *Probability and Finance: It's Only a Game,* that we were talking about this use of probability. No. We were talking about a new and very fundamental way of understanding how numerical probabilities arise and what they mean. I hope that the title of our new book, *Game-Theoretic Foundations*, will better convey what we are doing.

Applications to finance play a special role in our picture, because a financial market is an example of the simple game I have just described. A trader in the market plays the role of Skeptic, trying to multiply his money. The larger market plays both Forecaster (giving today's prices) and Reality (telling what Skeptic buys today is worth tomorrow). When the trader manages to multiply his money, he has called the market's efficiency into question. In mathematical finance, one usually adds probabilities for what the market will do to this picture. We show that this may not be necessary. Instead of putting probabilities into the game, we can use the hypothesis that a trader will not multiply his money relative to a market index to derive Ito's lemma and conclusions about the equity premium and other aspects of the behaviour of prices.

4. If there is one piece of information or advice that you would want your audience to take away from your book, what would that be?

Remember that when probabilities are numbers, they are about betting.

Everyone knows that the theory of mathematical probability began with calculations in games of chance. But that was long ago. Beginning with Jacob Bernoulli more

than three centuries ago, mathematicians and philosophers have tried to squeeze the betting out of probability. Instead of formalizing probability by defining betting games, they have formalized it with combinatorics or measure theory. Instead of understanding statistical tests as bets, with all the uncertainty this implies, they discuss the outcomes of these tests in terms of decisions and evidence, hoping to sound more objective and perhaps more certain. Bringing back the language of betting can help everyone understand what is really going on.

5. Who should read the book and why?

The book has something to say to everyone who develops and uses mathematical probability: pure mathematicians who are proving new theorems in probability, applied mathematicians who are using probability in statistics, machine learning, engineering, or business, and philosophers who are exploring the meaning of probability across the sciences.

The book is ideal for study at the graduate level. Each chapter has theorems and proofs, but most chapters also include a great deal of supplementary explanation. Each chapter has a section at the end entitled “Context”, which discusses the relation between the chapter’s ideas and similar ideas in the established theory of probability, in other fields, and in earlier centuries. Given the book’s breadth, from finance and statistics to physics, together with its broad view of abstract theory, from measure theory to imprecise probabilities, these “context” sections constitute an overview of mathematical probability that it would be difficult to put together from other sources.

6. Why, do you think, this book may be of interest now?

The book relates our theory to the established theory with a depth that we were not able to achieve in 2001. We now have purely game-theoretic proofs of Lévy’s zero-one law and many standard results in continuous time. In this sense, the book is essential to a full understanding of the established theory.

World-wide interest in game-theoretic probability itself has grown substantially since we published *Probability and Finance* in 2001. Workshops on the theory have been held and Britain, North America, and Japan. Many of the new results we report are due to a research group in Japan that was initiated by Kei Takeuchi and subsequently led by Akimichi Takemura.

The largest group of mathematicians now working on game-theoretic probability is in the area of mathematical finance. Many of these researchers call their work “probability-free”, because they do not add probabilities to the game of finance in the way the established theory does. The fourth part of our book, which is devoted to this topic, is the result of interaction between Professor Vovk, the Japanese school, and European students of probability-free finance. I expect that this part of the book will be foundational for further work in probability-free finance.

The abstract theory in the second half of the book shows that game-theoretic probability is also closely related to imprecise probability, a field of research that has become increasingly active across the world during the past twenty years. Much of the research now published under the rubric of imprecise probabilities could equally well be called research in game-theoretic probability. I believe that our book will be essential reading for students of imprecise probability.

I believe that the game-theoretic picture of probability is also poised to become important in mathematical statistics and philosophy. In statistics, testing-by-betting is poised to address the current acute dissatisfaction with p-values. On the philosophical side, the ``frequentist'' interpretation of probability is showing signs of wear, and a betting interpretation is well suited to deepen it. Objective probabilities are not always exactly frequencies. But in repetitive situations, we can bet on the frequencies coming close to them.

7. In addition to being a report on your research, your book is also a teaching resource. Exercises for the student are given at the end of each chapter. What led you to include exercises?

When we began writing the book, I did not believe that the subject was sufficiently mature for us to provide exercises for students. But as we worked, I realized that we had so much new to say that we needed to leave some of the details for the reader to work out. Because I still wanted to provide the reader with some guidance, I began to package these details as exercises. Once we began down this road, it was natural to add some simpler exercises to help students get started.

I hope the result will enable scholars interested in game-theoretic probability and its applications to include it in their teaching, even when they use other books as primary textbooks.

8. What is it about the area of game-theoretic probability and finance that fascinates you?

I'm not a young fellow; I started my doctoral study in mathematical statistics in 1970, and I've always been most interested in the philosophical and historical foundations of the subject. My interest in game-theoretic probability has evolved out of this philosophical interest. What does probability mean? Is it objective or subjective? Why do people say it is based on measure theory? How do we understand what statistical evidence is telling us?

My first book, in 1976, was concerned with measuring evidence. There I suggested, just as Jacob Bernoulli had suggested more than three centuries ago, that a numerical measure of the strength of evidence should not necessarily follow the usual rules for probabilities. Then what is the proper role for these rules? This question has occupied me since the 1980s. I believe that the game-theoretic picture provides the best setting for answering it.

9. What are you working on now?

There is a lot more mathematics to do within the game-theoretic framework, and I believe that this framework will also be important in the coming decades in many fields where probability is used. Some of the likely directions are indicated by the working papers at the website www.probabilityandfinance.com, which Professor Vovk and I have maintained since we published *Probability and Finance* in 2001.

At this moment, I am working on a paper that discusses how testing-by-betting can replace p-values. For the most part, however, I expect to be leaving relatively mathematical work on the game-theoretic picture to Professor Vovk and others, because I am anxious to complete a number of historical projects that I have set aside while completing our book.

The first historical project I hope to complete is my biography of Jean Ville, the French mathematician whose 1939 dissertation launched the mathematical notion of a martingale and inspired much of our thinking about game-theoretic foundations. My article on Ville's early years (The education of Jean André Ville, *Electronic Journal for the History of Probability and Statistics*, 5(1)) appeared in 2009, but his life during and after World War II is also fascinating and merits recounting. I also plan a book based on the article on Kolmogorov's foundations for probability that Professor Vovk and I published in 2006 (The sources of Kolmogorov's *Grundbegriffe*, *Statistical Science*,21:70-98) and another book on the game-theoretic foundations for probability advanced by Blaise Pascal and Christiaan Huygens.